Section Seven

A Reexamination of the Capital Asset Pricing Model

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7.1 INTRODUCTION

Much of the theoretical and most of the empirical research in modern capital market theory has been based on the Sharpe-Lintner-Mossin mean-variance, equilibrium model of exchange commonly called the Capital Asset Pricing Model (CAPM). However, the model has recently come under criticism from many sides. In an excellent survey article, Jensen [9] outlines the many controversies (both theoretical and empirical) and discusses a number of proposed alternatives. Briefly, the principal theoretical criticisms of the model have centered on (1) the assumption that investors choose their portfolios according to the Markowitz mean-variance criterion; (2) the perfect market assumptions; (3) the static or single-period nature of the model. On the empirical side, the main result of the CAPM that has been tested is the Security Market Line which specifies a relationship between the equilibrium expected return on an individual security to the expected return on the market portfolio. This relationship can be written as

$$E(\tilde{R}_i) = r + \beta_i[E(\tilde{R}_M) - r] \tag{7.1}$$

where $\tilde{R}_i$ is the random variable return on the i^{th} security; $\tilde{R}_M$ is the return on the market portfolio; r is the rate of interest; E is the expectation operator; $\beta_i \equiv \sigma_{iM} / \sigma_M^2$ with σ_{iM} equal to the covariance

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between $\tilde{R}_i$ and $\tilde{R}_M$ and σ_M^2 equal to the variance of $\tilde{R}_M$. Extensive testing by Friend and Blume [7] and Black, Jensen, and Scholes [2] seem to reject the hypothesis that assets are priced so as to satisfy Equation (7.1). In particular, the empirical security market line is too "flat." I.e., "low beta" ($\beta_i < 1$) securities have a larger return on average and "high beta" ($\beta_i > 1$) securities have a smaller return on average than is forecast by (7.1).

The approaches to the resolution of this controversy can be roughly classified into four groups: (1) those who believe that the CAPM is basically correct (or at least, that the security market line specification (7.1) does describe the relationship among expected returns) on a period-by-period basis and that the empirical discrepancies are caused by using the wrong index, missing assets, or by not taking into account that the (time-series for the) coefficients fluctuate in a stochastic manner (cf. Fama and MacBeth [6]); (2) those who believe that the CAPM is basically correct but that the perfect market assumptions are not. Hence, the model must be modified to allow for differentials between borrowing and lending rates, no short-sales, taxes, or the possibility of no riskless asset (cf. Black [1], Brennan [3]); (3) those who believe that due to intertemporal effects not considered in the one-year period CAPM, other sources of uncertainty besides market risk are significant in portfolio choice and hence the expected return on an asset will depend on more than its covariance with the market (cf. Merton [12]); (4) those who believe that the returns on assets are generated by a weighted linear combination of "common" or "systematic" factors plus an uncorrelated random term and that by "arbitrage" the expected return on an asset can be written in a general form similar to the Security Market Line specification, but that the CAPM itself has no relevance (cf. Ross [16]).

In Section 7.2 of this chapter an expositional discussion of the results in Merton [12] are presented and suggestions are made as to what other sources of uncertainty are likely to affect equilibrium expected returns on assets. In Section 7.3, a simple model is presented to provide an analytical framework for the discussion in Section 7.2.

Section 7-4 gives a brief discussion of why models of the type presented in Section 7-2 will, in general, give results different from the one-period, max-expected-utility-of-wealth models. It also explains the distinction between the "consumer services" model of Section 7-2 and the Ross [16] "Arbitrage-Factor" model.

7.2 A "CONSUMER SERVICES" MODEL OF ASSET PRICING

Most models of consumer choice postulate that each consumer acts

so as to:

$$\text{Max}\, E \quad U[C_1, C_2, \ldots, C_T; W_T] \tag{7.2}$$

where U is a well-behaved concave utility function; E is the expectation operator over the probability distributions of relevant random variables; $C_t = (C_{1t}, \ldots, C_{nt})$ is a vector bundle of consumption goods in period t of the consumer's life and end-of-life wealth; W_T, enters because of possible bequest motives. Since the cornerstone for much of economic theory is the assumption that all economic activities and institutions exist solely as the means to the ends of consumer satisfaction, a logical starting place for understanding why certain capital markets and financial securities exist is a study of the consumer-choice problem to determine the roles each of these markets or securities plays in helping consumers to maximize satisfaction. While this is certainly not a new idea, much of portfolio theory and capital market theory has been deduced based on the criterion of maximizing the expected utility of end-of-period wealth. While under certain conditions (cf. Fama [5]) (7.2) reduces to this type criterion, these condtions are rather specialized, and it will clarify much of the current controversy to return to the more basic criterion (7.2).

Even in the early work of Irving Fisher in a world of certainty and full information, it was recognized that the creation of financial securities and an exchange market for trading them would improve economic efficiency because endowments of individual economic units may not match optimal consumption plans in a temporal sense (i.e., financial securities make it possible to have savings by an individual economic unit not equal to investment). Moreover, the existence of such a market along with "bonds" for every maturity were sufficient for efficient intertemporal resource allocation as well as providing the appropriate "signals" for the efficient decentralization of the production and consumption decisions. Further, the return structure on all assets is completely determined by the prices of these bonds, and in particular, the one-period returns on all assets are equal to the one-period rate of interest.

When uncertainty is introduced, the problem became substantially more complicated. For the same efficiency to obtain, it is necessary to have securities and markets for every possible state of the world. While in such an Arrow-Debreu "complete markets" model the return structure on all assets is completely determined by the prices of these contingent claims, the model is so general that it is not empirically testable, and the enormous costs in running so many markets and processing the necessary information make its prescriptions economically infeasible. Because most of its theorems do not carry over for incomplete markets, care should be used in applying the model even as a "theoretical approximation." Nonetheless, in contrast to the CAPM which says that the consumer-investor requires only two financial securities (a one-period riskless bond and a mutual fund containing all assets in proportion to their value) for efficiency, the Arrow-Debreu model does demonstrate that financial securities may serve other roles for the consumer beyond that of providing an "efficient" risk-return tradeoff for end-of-period wealth.

Since the consumer does face relevant uncertainties in addition to the uncertainty of his end-of-period wealth, the natural inclination is to develop a model which explicitly takes into account the effects of these other uncertainties on asset pricing (as does the Arrow-Debreu model) while at the same time, introducing enough structure and restrictions to the model to give it the same analytical simplicity and empirical tractability as the CAPM. In [12], an equilibrium model of this type was developed based on intertemporal utility maximization in continuous-time where the uncertainties were described by diffusion-type stochastic processes. However, because of the rather specialized technical tools required for that analysis, an expositional development of the basic results of that model along with some extensions would seem appropriate.

We start with the Arrow-Debreu model with complete markets where there are more securities[1] (N) than states of nature (n). Cass and Stiglitz [4] have proved a "mutual fund" or "separation" theorem which states that there can be constructed n mutual funds or (composite) securities made up of linear combinations of the N securities such that (1) all consumer-investors would be indifferent between having available just these n mutual funds or all the N securities; (2) the construction of these composite securities requires no knowledge of consumer preferences, wealth allocations, or their subjective probabilities for each of the states of nature.

While the theorem states "indifference" when there are no transactions costs and information is freely available to everyone, it is reasonable to presume strict preference for the mutual funds if $N >> n$. Economies of scale in transactions costs and information gathering and processing make it more sensible to have a centralized compilation of the distributions for each of the N securities rather than have each investor do it for himself. For the same reason, it would make sense to have each mutual fund have the property of paying a positive amount in one state of nature and zero otherwise (i.e., basic contingent claims) rather than some other more complicated combination which in theory would be equivalent.

Since the number of possible states of nature is very large, such a complete set of markets is economically unfeasible. There are three basic reasons: (1) the direct costs of operation of so many separate mutual funds; (2) despite the reduction from N to n, the large size of n would make the consumer's information processing costs very large; (3) the occurrence of certain states may be controllable by some consumers (i.e., the moral hazard problem). Hence, some "compromise" is obviously required. To do so, we retain the notion that the mutual fund approach is preferred when there are large numbers of securities and large numbers of relatively small economic units (e.g., consumers), but we restrict the number of funds. Consumers have access to limited amounts of information and have limited abilities to process the information that they have. Further, because of the costs of information gathering and processing, one would expect the consumer to center his attentions on the major sources of uncertainty which affect his consumption plan. Therefore, it is reasonable to assume that these sources of uncertainty can be represented by a finite and not very large number of state variables. One would expect the number and type of mutual funds[2] that would be created to correspond roughly to the number and type of major uncertainties which consumers face. The primary prerequisites for such a fund to be created are: (1) the source of uncertainty must be important to a sufficient number of consumers; (2) it must be possible to have a standardized contract with payoffs in contingencies which are easily recognizable; (3) the source of uncertainty must not be controllable by the consumer(s). Thus, for some of the major uncertainties, it is virtually impossible to construct a financial security which would allow the consumer to hedge against them. Broadly, we would expect to find two types of securities traded: (1) "natural" securities such as common stocks which are issued by firms to finance production of real output; and (2) financial securities or financial intermediaries created to serve the purposes of the "mutual funds" discussed above. These purposes are to aid the consumer in achieving a higher (expected) consumption level (through a return to capital), a better intertemporal allocation of resources (by not requiring that savings equals investment at each point in time), and a lower level of risk (by providing hedges against the major (common) sources of uncertainty faced by the consumer).

The individual consumer's demand for assets can be categorized as follows: for those important sources of uncertainty which he faces in common with other consumers, he will take positions in the mutual funds or financial securities created for that purpose. For those important sources of uncertainty for which no mutual fund exists (either because it is specialized to him or there is inherent moral hazard), he will take positions in those primary securities, if they exist, to hedge. For those sources of uncertainty for which no security is a hedge and for those sources which he neglects in his analysis, no security position can help so his (differential) demand for securities will be unaffected.

As an example, if the consumers are one-period maximizers of the utility of mean and variance of end-of-period wealth, then the well-known separation or two-fund theorem obtains. Namely, each investor will be indifferent between selecting a portfolio from all the primary securities and from two funds: (1) the market portfolio of risky assets and (2) a riskless asset. Presumably, such funds would be created since there is an obvious common demand. However, suppose one investor also has (uncertain) labor income which he cannot sell forward to eliminate its risk because of the moral hazard problem. Suppose further that it is a very highly-specialized form of labor which can only be used by one (or a small number of similar) company. Then, from risk-aversion, it would be natural to suppose that this investor would want to hold less of this company's stock than is represented in the market portfolio. Hence, in addition to the two mutual funds, he would want to short-sell some amount of this particular company's stock to hedge against unfavorable changes in his labor income. If on the other hand, there was no security whose outcome was correlated with this particular source of uncertainty, his optimal portfolio would be generated by the two mutual funds alone.

Thus, to a reasonable approximation, most of the aggregate demand for the individual primary securities can be viewed as coming in an "indirect way through the mutual funds." I.e., individual consumers for the most part only purchase a relatively small number of composite financial securities or mutual funds. Mutual fund managers purchase the primary securities to form the portfolios necessary to perform these services. Therefore, the aggregate demand for a primary security will depend on how its return contributes to the formation of these "service" portfolios.

Since the equilibrium expected return on an asset is "determined" by the aggregate demand for it, one would expect to find a correspondence between its expected return and the statistical dependence between the asset's return and the various major sources of uncertainty. All risk-averse consumers would prefer to have less uncertainty for the same expected consumption stream, and would "give up" some (expected) return on an asset in return for that asset

providing a hedge against some of these uncertainties. Hence, to the extent that any asset's return contributes to (or aggravates) the consumers' attempts to hedge against these uncertainties, one would expect the equilibrium return on that asset to be affected. If, on average, a particular asset's return contributes to consumers' attempts to hedge against a common source of uncertainty, then one would expect that the equilibrium expected return on that asset to be differentially lower than on a similar asset which does not provide that "service." This negative differential in expected return can be interpreted as the market "cost" to the consumer for the hedging service provided by this asset. If, on average, an asset's return would aggravate consumers' attempts to hedge, then the equilibrium expected return would be differentially higher, and this positive differential in expected return can be interpreted as the market "premium" to the consumer in return for bearing the extra risk caused by holding this asset. A simple illustration of this principle can be found in the CAPM. Since the only source of uncertainty is end-of-period wealth and all investors are assumed to be risk-averse, a given investor would view an asset as providing a ("diversification") service if it lowers the variance of his end-of-period wealth and, hence, would accept a lower expected return on this asset than on one which did not provide this service. However, since all investors' optimal portfolios are perfectly-correlated, an asset which aids diversification for one investor does so for all investors, and therefore, all investors would accept a lower expected return on this asset. Inspection of the security market line, (7.1), shows this is the case.

However, with respect to most sources of uncertainty, such unanimity among investors' views of whether an asset contributes to risk or not will be the exception. Thus, one group of consumers may consider a *long* position in an asset as contributing to a reduction of the risks it perceives while another group may view a *short* position as contributing to a reduction in its risks. Thus, whether the market expected return on the asset represents a differential cost or premium will depend on the aggregation of investor's demands, and unless there is a systematic "weak side" to the market, the sign of the differential may fluctuate through time. One example of this type is the Modigliani-Sutch [13] Habitat theory of bond pricing. If a consumer has preferences which induce risk-aversion with respect to wealth, then he will view long-term bonds as risky and would require a market premium over short-term bonds to hold them. If a consumer has preferences which induce risk-aversion with respect to income, then short-term bonds are risky to him, and he would require a market premium over long-term bonds to hold them. Thus, with respect to the uncertainty about future interest rates, the differential expected return between long- and short-term bonds could be of either sign.

To determine the types of securities one would expect to find and the sources of differentials in expected returns, it is necessary to establish what the important uncertainties are facing a typical consumer making a plan according to the criterion in (7.2). Although not a complete listing, the following seven items would seem to cover most of the important common sources of uncertainty for consumers:

(S.1) uncertainty about his own future tastes;
(S.2) uncertainty about the menu of possible consumption goods that will be available in the future;
(S.3) uncertainty about relative prices of consumption goods;
(S.4) uncertainty about his labor income;
(S.5) uncertainty about future values of nonhuman assets;
(S.6) uncertainty about the future investment opportunity set, i.e., the future rates of return which can be earned on capital;
(S.7) uncertainty about the age of death.

While all of these have probably been considered in one model or another, it is important to note that all models which use the criterion of maximizing the expected utility of end-of-period wealth explicitly take into account only the uncertainty in (S.5). Included in this class of models is the CAPM.

Even though all these uncertainties are important to the consumer, not all will differentially affect security prices or returns. It is difficult to image a financial security which could reduce the uncertainties associated with (S.1) or (S.2). While (S.7) is an important problem for all consumers and life insurance was created in response to this demand, the event of death is probably reasonably statistically independent among people, and it is unlikely that the returns on securities (other than life insurance policies) would be statistically dependent on the event of an individual's death. Hence, one would not expect (S.7) to cause differential effects on security prices. The risks associated with (S.4) could be completely eliminated if the consumer could sell forward his wage income in the same way shares are issued on nonhuman capital. Because of the moral hazard problem, it is difficult for the consumer to sell forward his wage income. While some of the individual risk can be eliminated by disability and life insurance and by "investing" in education to make his labor more substitutable across firms, there still will be systematic risk due to (unanticipated) shifts in capital and labor's relative shares (i.e., the wage-rental ratio). This could produce a differential demand for shares in labor-intensive versus capital-intensive industries. Inflation risk (S.3) may cause differentials in demand between different maturity "money" securities. Although information costs and the uncertainties (S.1) and (S.2) prohibit complete future markets for consumption goods, it is reasonable to expect consumers to differentiate broad classes of consumption (e.g., housing, food, transportation, clothing, and recreation) and hence, differentials in demand for shares in different industries could occur as the result of (S.3). (S.5) is the standard end-of-period wealth uncertainty and hence, differential demands will occur for securities which aid diversification. Finally, as is discussed in [12], if there is uncertainty about the rates of return which will be available in the future, differential demands may occur between long and short-term bonds and between shares of firms whose returns are sensitive to shifts in capitalization rates versus ones that are not.

If these are the sources of uncertainty common to most investors, then we can identify a set of mutual funds which would be (approximately) sufficient to span the space of consumers' optimal portfolios. Specifically, we might identify these funds to be: (1) the "market" portfolio; (2) a (short-term) riskless asset; (3) hedging portfolios for *unanticipated* shifts in rates of return; (4) shifts in the wage/rental ratio; and (5) changes in prices for basic groups of consumption goods. Further, consumer demand for individual securities can be written as if they came indirectly through the demands for these mutual funds. Hence, the equilibrium expected return on a security will be a function of the expected return on each of these funds and the statistical dependence between the security's return and the return on each of these funds.

In the special case of continuous-trading examined in Merton [12], the equilibrium expected return on the k^{th} security satisfies

$$E(\tilde{R}_k) - r = \sum_{i=1}^{m} \beta_{ik} [E(\tilde{R}_{im}) - r] \qquad (7.3)$$

where $\tilde{R}_{im}$ is the return on the i^{th} mutual fund and β_{ik} is the instantaneous multiple regression coefficient between the return on the k^{th} security and the return on the i^{th} mutual fund, and m is the number of mutual funds necessary to span the space of optimal portfolios.

To empirically test the model, it may be necessary to construct portfolios (in a fashion similar to the Black, Jensen, and Scholes [2] method for the "zero-beta" portfolio) which have the properties of the hypothesized mutual funds when no such portfolio already exists. Further, the specification (7.3) does not rule out changes over time in the $\{\beta_{ik}\}$ or the $E(\tilde{R}_{im})$. Hence, care must be taken in choosing sufficiently small observation intervals to avoid (or at least limit) the nonstationarity problem.

7.3 A SIMPLE EXAMPLE

Consider a consumer-investor who lives for two periods and consumes only at the end of the second period. At the beginning of period one (time zero), he receives a wage income of $y(0)$. This and his initial wealth, $W(0)$, are then allocated in a portfolio among four assets: shares in two firms, a two-period discount bond, which pays $1 at the end of period two with certainty, and a riskless one-period bond. At the beginning of period two (time one), he receives a wage income of $y(1)$. This and his wealth at that time, $W(1)$, are then allocated in a portfolio among three assets: the shares in the two firms and a riskless one-period bond. At the end of period two (time two), he allocates his wealth among two consumption goods so as to maximize a strictly concave utility function of consumption.

The uncertainties that he faces as of time zero are in addition to uncertainty of returns on the risky assets, uncertainty about next period's wage income, next period's rate of interest, and the end-of-period price of consumption good number two. Consumption good number one is numeraire, and hence, by definition, its price is always one.

We postulate a very simple set of stochastic processes for the change in wage income, consumption prices, and interest rates. Namely,

$$\tilde{P}(t+1) - P(t) = v_2\tilde{Z}_2(t) \tag{7.4a}$$

$$\tilde{y}(t+1) - y(t) = v_1\tilde{Z}_1(t) \tag{7.4b}$$

$$\tilde{r}(t+1) - r(t) = -v_3\tilde{Z}_3(t) \tag{7.4c}$$

where $P(t)$ is the price of good two at time t; $y(t)$ is the wage income at time t; $r(t)$ is the interest rate at time t. Further, it is assumed that the v_i are constants and

$$E\left\{\tilde{Z}_i(t)\right\} = 0 \quad \text{for all } i \text{ and } t, \tag{7.5a}$$

$$E\left\{\tilde{Z}_i^2(t)\right\} = 1 \quad \text{for all } i \text{ and } t, \tag{7.5b}$$

$$E\left\{\tilde{Z}_i(t)\tilde{Z}_j(t)\right\} = 0 \quad \text{for } i\neq j \text{ and all } t, \tag{7.5c}$$

$$E\left\{\tilde{Z}_i(t)\tilde{Z}_j(t+\tau)\right\} = 0 \quad \text{for all } i, j \text{ and } \tau\neq 0 \tag{7.5d}$$

Thus, the expected change in these variables is zero and the changes have zero serial and cross-sectional correlations.

The return structure on the assets can be written as

$$\tilde{R}_1(t) = r(t) + \alpha_1 + a_1\tilde{Z}_1(t) + \tilde{\epsilon}_1(t) \tag{7.6a}$$

(firm #1)

$$\tilde{R}_2(t) = r(t) + \alpha_2 + a_2\tilde{Z}_2(t) + \tilde{\epsilon}_2(t) \tag{7.6b}$$

(firm #2)

$$\tilde{R}_3(t) = r(t) + \alpha_3 + a_3\tilde{Z}_3(t) \tag{7.6c}$$

(two-period bond)

where α_i and a_i are constants with $a_i > 0$

and

$E\{\tilde{\epsilon}_i(t)\} = 0$; $E\{\tilde{\epsilon}_i(t)\tilde{\epsilon}_j(t)\} = 0$ for $i \neq j$;

$E\{\tilde{\epsilon}_i(t)\tilde{Z}_j(t)\} = 0$ for all i, j;

$E\{\tilde{\epsilon}_i(t)\tilde{\epsilon}_j(t+\tau)\} = E\{\tilde{\epsilon}_i(t)\tilde{Z}_j(t+\tau)\} = 0$ for all i, j and $\tau \neq 0$;

Variance $(\tilde{R}_i(t)) \equiv \sigma_i^2$.

Thus, the return on firm #1 is positively correlated with changes in wage income; the return on firm #2 is positively correlated with changes in the price of consumption good two; the return on the two-period bond is (perfectly) negatively correlated with changes in the interest rate.

To solve the consumption-investment problem, we use the standard method of stochastic dynamic programming (cf. Hakansson [8] or Samuelson (14]) which requires us to start at the end of the program and work backwards.

At time two, the investor will know his wealth, $W(2)$, and the price of consumption good two, $P(2)$, and will act so as to

$$\underset{\{C_1, C_2\}}{\text{Max}} \quad U[C_1, C_2] \tag{7.7}$$

subject to the budget constraint: $W(2) = C_1 + P(2)C_2$, where $U[C_1, C_2]$ is a strictly concave utility function. The first-order condition to be satisfied by the optimal choice, $\{C_i^*\}$, is

$$\frac{\partial U}{\partial C_1} \Big/ \frac{\partial U}{\partial C_2} = 1/P(2) \tag{7.8}$$

From (7.8), we can solve for $C_i^* = C_i^*[W(2), P(2)]$. Define the indirect utility function, ϕ, to be

$$\phi[W(2), P(2)] \equiv U[C_1^*, C_2^*] \tag{7.9}$$

As time one, the investor will know his wealth (which includes his period-one wage income, $y(1)$), $W(1)$, the price of consumption good two, $P(1)$, and the interest rate, $r(1)$. Since he does no consuming at that time, his only decision is to choose a portfolio allocation $\{w_1(1), w_2(1), w_3(1)\}$ so as to

$$\text{Max}\, E_1 \left\{\phi[\tilde{W}(2), \tilde{P}(2)]\right\} \tag{7.10}$$

Subject to the constraint $\Sigma_1^3 w_i(1) = 1$ where $w_i(1)$ is the fraction of his wealth invested in the i^{th} asset and "E_1" is the conditional expectation operator, conditional on knowing all (relevant) events as of time one. We can write end-of-period wealth, $W(2)$, in terms of the decision variables as

$$\begin{aligned}\tilde{W}(2) &= W(1) + W(1)[\Sigma_1^2 w_i(1)[\tilde{R}_i(1) - r(1)] + r(2)] \\ &= W(1) + W(1)[\Sigma_1^2 w_i(1)\alpha_i + r(1)] \\ &\quad + W(1)\left\{\Sigma_1^2 w_i(1)a_i\tilde{Z}_i(1) + \Sigma_1^2 w_i(1)\tilde{\epsilon}_i(1)\right\}\end{aligned} \tag{7.11}$$

where the constraint has been substituted out (i.e., $w_3(1) = 1 - \Sigma_1^2 w_i(1)$) and $\tilde{R}_i(1)$ has been substituted for from (7.6).

To provide explicit solutions and to relate them to the standard mean-variance analysis, we make a quadratic approximation of the type described in Samuelson [15] and justified for "short-time intervals" in Merton [10,11,12]. Namely,

$$\begin{aligned}E_1\left\{\phi[\tilde{W}(2), \tilde{P}(2)]\right\} &\doteq E_1\Big\{\phi[W(1), P(1)] + \phi_1[W(1), P(1)]\,\Delta\tilde{W} \\ &\quad + \phi_2[W(1), P(1)]\,\Delta\tilde{P} + 1/2[\phi_{11}(\Delta\tilde{W})^2 \\ &\quad + 2\phi_{12}(\Delta\tilde{W}\Delta\tilde{P})\phi_{22}(\Delta\tilde{P})^2]\Big\}\end{aligned} \tag{7.12}$$

where subscripts denote partial derivatives and "Δ" denotes change over the period. From (7.4), (7.5), (7.6), and (7.11), we have that

$$E_1[\Delta\tilde{W}] = W(1)[\Sigma_1^2 w_i(1)\alpha_i + r(1)], \tag{7.13}$$

$$E_1[\Delta\tilde{P}] = 0,$$

and

$$\begin{aligned}E_1\left\{(\Delta\tilde{W})^2\right\} &= W^2(1)\left\{(\Sigma_1^2 w_i(1)\alpha_i + r(1))^2 + \Sigma_1^2 w_i^2(1)\sigma_i^2\right\} \\ &\doteq W^2(1)\,\Sigma_1^2 w_i^2(1)\sigma_i^2,\end{aligned} \tag{7.14a}$$

$$E_1\left\{(\Delta\tilde{P})^2\right\} = v_2^2, \tag{7.14b}$$

$$E_1\left\{\Delta\tilde{W}\Delta\tilde{P}\right\} = W(1)w_2(1)a_2v_2 \tag{7.14c}$$

where the approximation in (7.14a) is valid for short-time intervals since asymptotically, $(\Sigma^2 w_i(1)\alpha_i + r(1))^2 << \Sigma_1^2 w_i^2(1)\sigma_i^2$. Taking the expectation term-by-term in (7.12) (and noting that because $\phi[\tilde{W}, \tilde{P}]$ and its derivatives are evaluated at the $[W(1), P(1)]$, they are nonstochastic) and substituting from (7.13) and (7.14), we can rewrite (7.12) as

$$E_1\left\{\phi[\tilde{W}(2),\tilde{P}(2)]\right\} \doteq [\phi + 1/2\phi_{22}v_2^2 + \phi_1 W(1)[\Sigma_1^2 w_i(1)\alpha_i + r(1)]$$

$$+ \phi_{12}W(1)w_2(1)a_2v_2$$

$$+\frac{W^2(1)}{2}\phi_{11}\Sigma_1^2 w_i^2(1)\sigma_i^2]. \qquad (7.15)$$

The first-order conditions for an interior maximum in (7.15) are

$$0 = \phi_1 W(1)\alpha_1 + W^2(1)\phi_{11}w_1^*(1)\sigma_1^2 \qquad (7.16a)$$

and

$$0 = \phi_1 W(1)\alpha_2 + \phi_{12}W(1)a_2v_2 + W^2(1)\phi_{11}w_2^*(1)\sigma_2^2 \qquad (7.16b)$$

Solving (7.16) for the optimal demands for risky assets, $d_i^*(\equiv w_i^*(1)W(1))$, we have that

$$d_1^*(1) = A\left(\frac{\alpha_1}{\sigma_1^2}\right) \qquad (7.17a)$$

$$d_2^*(1) = A\left(\frac{\alpha_2}{\sigma_2^2}\right) + H\left(\frac{a_2v_2}{\sigma_2^2}\right) \qquad (7.17b)$$

where

$$A \equiv -\phi_1[W(1),P(1)]/\phi_{11}[W(1),P(1)] \qquad (7.18a)$$

and

$$H \equiv -\phi_{12}[W(1),P(1)]/\phi_{11}[W(1),P(1)] \qquad (7.18b)$$

Note that unlike in the standard mean-variance, the ratio of $d_1^*(1)/d_2^*(1)$ is not independent of preferences (unless $H \equiv 0$), and hence, the separation theorem does not obtain. Therefore, all investors will not hold the same (relative) proportions of risky assets. The reason for this is that due to the end-of-period price uncertainty between the consumption goods, the indirect utility function depends on other variables in addition to end-of-period wealth.

Substituting from (7.17) into (7.15), we can define a new "indirect" utility function

$$\psi[W(1),P(1),r(1)] \equiv \text{Max } E_1 \quad \phi[\tilde{W}(2),\tilde{P}(2)] \qquad (7.19)$$

subject to the constraint $\Sigma_1^4 w_i(0) = 1$. End-of-period wealth can be written as

$$\tilde{W}(1) - W(0) = \tilde{y}(1) + W(0)[\Sigma_1^3 w_i(0)\alpha_i + r(0)]$$

$$+ W(0)\left\{\Sigma_1^3 w_i(0)a_i\tilde{Z}_i(0) + \Sigma_1^2 w_i(0)\tilde{\epsilon}_i(0)\right\} \qquad (7.20)$$

As was done in (7.15), we can use a similar quadratic expansion for ψ by expanding around $[W(0)+y(0),P(0),r(0)]$. Namely,

$$E\left\{\psi[\tilde{W}(2),\tilde{P}(1),\tilde{r}(1)]\right\} \doteq E_0\left\{\psi + \psi_1\Delta\tilde{W} + \psi_2\Delta\tilde{P} + \psi_3\Delta\tilde{r}\right.$$

$$+ 1/2[\psi_{11}(\Delta\tilde{W})^2 + \psi_{22}(\Delta\tilde{P})^2$$

$$+ \psi_{33}(\Delta\tilde{r})^2 + 2(\psi_{12}(\Delta\tilde{W}\Delta\tilde{P})$$

$$\left.+ \psi_{13}(\Delta\tilde{W}\Delta\tilde{r}) + \psi_{32}(\Delta\tilde{P}\Delta\tilde{r}))]\right\} \qquad (7.21)$$

Taking expectations term-by-term, substituting from (7.4), (7.5), (7.6), and (7.20), and eliminating terms which do not involve the decision variables, we have that maximizing (7.21) is equivalent to

$$\underset{\{w_1,w_2,w_3\}}{\text{Max}}\left\{\psi_1 W(0)(\Sigma_1^3 w_i(0)\alpha_i + r(0))\right.$$

$$+ 1/2\psi_{11}[(\Sigma_1^3 w_i^2(0)\sigma_i^2)W^2(0) + 2v_1a_1w_1(0)W(0)]$$

$$\left.+ \psi_{12}W(0)w_2(0)a_2v_2 - \psi_{13}W(0)w_3(0)a_3v_3\right\} \qquad (7.22)$$

The first-order conditions for an interior maximum are

$$0 = \psi_1\alpha_1 + \psi_{11}[w_1^*(0)\sigma_1^2 W(0) + a_1v_1] \qquad (7.23a)$$

$$0 = \psi_1 \alpha_2 + \psi_{11} [w_2^*(0) \sigma_2^2 W(0)] + \psi_{12} a_2 v_2 \tag{7.23b}$$

$$0 = \psi_1 \alpha_3 + \psi_{11} [w_3^*(0) \sigma_3^2 W(0)] - \psi_{13} a_3 v_3 \tag{7.23c}$$

Solving (7.23) for the optimal demands for risky assets, $d^* (\equiv w_i^*(0)W(0))$, we have that

$$d_1^*(0) = A\left(\frac{\alpha_1}{\sigma_1^2}\right) - \frac{a_1 v_1}{\sigma_1^2} \tag{7.24a}$$

$$d_2^*(0) = A\left(\frac{\alpha_2}{\sigma_2^2}\right) + H\left(\frac{a_2 v_2}{\sigma_2^2}\right) \tag{7.24b}$$

$$d_3^*(0) = A\left(\frac{\alpha_3}{\sigma_3^2}\right) + K\left(\frac{a_3 v_3}{\sigma_3^2}\right) \tag{7.24c}$$

where $A \equiv -\psi_1/\psi_{11}$, $H \equiv -\psi_{12}/\psi_{11}$, and $K \equiv +\psi_{13}/\psi_{11}$. Note that the portfolio demands for the "usual," one-period mean-variance maximizer would be $A\alpha_i/\sigma_i^2$. However, the multiperiod nature of the plan coupled with price, interest rate, and (future) wage income uncertainty results in differential demands and since $d_i^*(0)/d_j^*(0)$ are not independent of preferences, the separation theorem does not obtain. These differential demands can be interpreted as "hedging" demands induced by uncertainties about important variables other than terminal wealth. Note, for example, that because the return on security number one is positively correlated with future wage income, the investor holds less of this security than he would have if he had no wage income.

While a full discussion and derivation of the equilibrium relationship among expected returns in the presence of these additional uncertainties can be found in Merton [12], we can easily demonstrate the failure of the classical security market line (7.1) for the case leading to the demand functions in (7.24). Suppose that the aggregate supply of two-period bonds is zero (i.e., all such debt is "inside" debt). Then, in equilibrium, the aggregate demand for bonds must be zero. Since by construction, the returns on these bonds are uncorrelated with all other assets, these two conditions imply that these bonds will have a "zero beta" (zero covariance with the market). By the CAPM, the equilibrium expected return, $E(\tilde{R}_3)$, must equal the riskless rate, r, or in other words, $\alpha_3 = 0$. By summing Equation (7.24c) over all investors with $\alpha_3 = 0$, we have that the aggregate demand for the bonds, D_3, will equal

$$D_3 \equiv \Sigma d_3^j(0) = \left(\frac{a_3 v_3}{\sigma_3^2}\right) \Sigma K^j \tag{7.25}$$

where j denotes the j^{th} investor. Unless $\Sigma K^j \equiv 0, D_3 \neq 0$, and hence $\alpha_3 = 0$ will not be the equilibrium expected excess return. Thus, one should not expect the security market line relationship to obtain except in very specialized cases. However, a generalized security market plane of the type specified in (7.3) might well provide an adequate description of equilibrium expected returns.

7.4 CONCLUSION

As illustrated in the example in the previous section, the consumer services model leads to demand functions and an equilibrium structure of returns for assets that are fundamentally different from those derived in models where investors maximize the expected utility of end-of-period wealth. While the stochastic dynamic programming technique produces a "derived" utility function of end-of-period wealth[3] whose expected value is maximized at each stage in the program, this "derived" utility function is not only a function of wealth but also of the other state variables of the problem (e.g., in the example of Section 7.3, these variables included relative consumption good prices and interest rates). Thus, the portfolio combination that maximizes the expected value of this utility function might be one that would never be chosen by a maximizer of expected utility of wealth. I.e., the optimal portfolio could be "inefficient" as measured by the usual methods of stochastic dominance. Moreover, these differences in the optimal portfolio holdings will appear unless the marginal utility with respect to wealth does not depend on the other state variables or unless the distribution of returns on assets is independent of the other state variables.

The specific linear structure of equilibrium expected returns described at the end of Section 7.2 depends on the validity of the local quadratic approximation of the derived utility function at each stage in the dynamic program. As mentioned in the example of Section 7.3, such an approximation is valid if the length of time between successive decisions is "small" and becomes more accurate as the asymptotic limit of continuous trading is approached. Moreover, I know of no other assumption justifying this approximation that is not severely at variance with economic facts (e.g., quadratic utility globally or gaussian-distributed returns). The Ross "Arbitrage Factor" model deduces a similar linear equilibrium structure of expected returns without the explicit assumption of the quadratic approximation. However, this model assumes that the stochastic returns on assets are generated in a linear fashion by one or more exogenous, stochastic factors. Careful analysis shows that such a specification can only be a reasonable approximation to economic reality if the time interval between successive changes is small as well. Hence, as in the usual mean-variance model, this model requires the assumption of (approximate) continuous trading as a necessary condition for its economic validity.

NOTES TO SECTION SEVEN

1. It is assumed that there is at least one security with a positive payoff in at least one of each of the states of nature.
2. "Mutual fund" is used here in the broad sense including financial security or financial intermediary.
3. More generally, in the literature of optimal control, this function is called the Bellman function. The ϕ and ψ functions of the example in Section 7.3 are examples of this derived utility function.

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